$$\Rightarrow \lim_{n\to\infty} \frac{\left(\sum_{k=2}^{n}\left(\left(\frac{k}{e}\right)^{k}\sqrt{2\pi k}\right)^{\frac{1}{n}}\right)^{\frac{1}{n}}}{\left(\frac{\left(\frac{2n+1}{e}\right)^{2n+1}\sqrt{n+2}\sqrt{\pi}}{2^{n-1}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}}\right)^{\frac{1}{n+1}}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=2}^{n}\left(\frac{k}{e}\right)\cdot\left(\sqrt{2\pi k}\right)^{\frac{1}{n}}}{\left(\frac{\left(\frac{2n+1}{e}\right)^{\frac{2n+1}{n+1}}\left(\sqrt{n+2}\right)^{\frac{1}{n+1}}}{2^{n-1}\left(\frac{n}{e}\right)^{\frac{n}{n+1}}\left(\sqrt{2n}\right)^{\frac{1}{n+1}}}\right)}$$

As we can see many terms are cancelling and as we can see,

$$\frac{2n+1}{n+1} = \frac{2+\frac{1}{n}}{1+\frac{1}{n}} \to 2 \ as\ n \to \infty$$

And similarly,

$$\frac{n}{n+1} \to 1,\ as\ n \to \infty$$

Then, we apply the ratio test, we get, our limit $L \to 0$
Hence,

$$L = 0$$

(Answer)

Shivam Sharma

Forth solution. First note that $\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}} = 1$. Indeed, we have

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{(2n+1)!!}}{\sqrt[n+1]{(2n+1)!!}} \cdot \sqrt[n]{\frac{(2n-1)!!}{(2n+1)!!}} = \lim_{n\to\infty} \frac{\sqrt[n]{(2n+1)!!}}{\sqrt[n+1]{(2n+1)!!}} \cdot \frac{1}{\sqrt[n]{2n+1}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{(2n+1)!!}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \sqrt[n(n+1)]{(2n+1)!!}$$

and since by AM-GM Inequality

$$\sqrt[n+1]{(2n+1)!!} < \frac{1+3+\ldots+2n+1}{n+1} = \frac{(n+1)^2}{n+1} = n+1$$

then $\sqrt[n(n+1)]{(2n+1)!!} < \sqrt[n]{n+1}$.
Noting that $\lim\limits_{n\to\infty} \sqrt[n]{n+1} = 1$ and $\sqrt[n]{(2n+1)!!} > 1$, by Squeeze Principle we obtain

$$\lim_{n\to\infty} \sqrt[n(n+1)]{(2n+1)!!}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \sqrt[n]{\frac{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}}{(2n-1)!!}} \cdot \frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \sqrt[n]{a_n},$$

where

$$a_n := \frac{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}}{(2n-1)!!}.$$

Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}{(2n+1)!!}}{\dfrac{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}}{(2n-1)!!}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{2n+1} = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{2n+1} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{e} \cdot \frac{1}{2} = \frac{1}{2e}$$

then by Cauchy's second theorem on limits we obtain that

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\sqrt[3]{3!}\ldots\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}} = \frac{1}{2e}. : 0.183\,94$$

Arkady Alt